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ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

By MR. R. S. WOODWARD, Washington, D. C.

1. A considerable class of problems concerning the sea-level is that in which the attracting or disturbing mass is symmetrically disposed about a radius of the earth's surface and is situated on or near that surface. Examples of such masses, proximate at least, are portions of the earth's crust, like a continental table-land, and the ice-caps which are supposed to have been of great extent and thickness during certain epochs of the earth's history.

It is proposed in the following paper to develop the theory of the solution of this class of problems so far as is necessary to render practicable the numerical evaluation of the characteristic effects of the disturbing mass in any special case.

2. The solution of all problems of the kind outlined above depends on the principle of hydrostatics that the potential of the forces producing a free liquid surface in equilibrium has a constant value for all points of that surface. In the case of the earth, if the potential of all the attractive forces acting on a unit mass at any point of the sea-surface be denoted by P, the distance of the point from the axis of rotation by l and the velocity of rotation by ω, the form of the surface will be completely defined by the equation

$$P + \tfrac{1}{2}l^2\omega^2 = \text{a constant.} \qquad (1)$$

The surface defined by this equation is now commonly called the *geoid*. It, as represented by the ocean surface, is a real surface and does not necessarily coincide with the earth's ellipsoid, which is an ideal surface of reference.

3. The exact value of P in (1) is a complicated function of the densities of the element-particles of the earth and of the co-ordinates of those particles and the attracted point. For the present purposes, however, it will suffice to consider P due to a centrobaric sphere of equal mass and volume with the earth

and concentric with the earth's centre of gravity. Since we shall only deal with relative positions of any point on the sea-surface, the potential due to centrifugal force, which is represented by the second term in (1), may be neglected.

If a and b denote the equatorial and polar semi-axes respectively of the earth's ellipsoid, and r_0 the radius of the sphere just referred to,

$$r_0 = \sqrt[3]{a^2b}. \tag{2}$$

The known values of a and b (Clarke, 1866) give $r_0 = 20902394$ English feet, or in round numbers 21000000 feet. The surface of the sphere thus defined may be regarded as the surface assumed by a thin film of sea-water covering a nucleus whose mass plus the mass of the film equals the earth's mass. We shall call this ideal surface the undisturbed surface. With respect to it the real surface of the earth lies partly without and partly within; but, so far as small relative changes in sea-level are concerned, it is practically immaterial whether we refer to the actual, closely spheroidal surface or to the simpler spherical one.

4. Let $M =$ mass of the earth,
$\rho_m =$ mean density of the earth.

Then

$$M = \tfrac{4}{3}r_0^3\pi\rho_m; \tag{3}$$

and the equation to the undisturbed surface is

$$\frac{M}{r_0} = \tfrac{4}{3}r_0^2\pi\rho_m = C_1, \text{ a constant.} \tag{4}$$

Suppose now a new mass m, of density ρ (positive or negative), placed in any fixed position relative to the undisturbed surface. The resulting new sea-surface will then differ from that defined by (4). To determine this difference, let V be the potential of the disturbing mass m at any point of the disturbed surface, and let v denote the elevation or depression of this point with respect to the undisturbed surface. The equation of the disturbed surface will then be

$$\frac{M}{r_0 + v} + V = C_2, \text{ a constant.} \tag{5}$$

The difference of this and (4), to terms of the first order inclusive in v, is

$$-\frac{M}{r_0^2}v + V = C_2 - C_1;$$

whence, putting

$$V_0 = C_2 - C_1,$$

we have

$$v = (V - V_0)\frac{r_0^2}{M}. \tag{6}$$

Since (if we omit the unit of attraction, which disappears by division from

our expression) $M/r_0^2 = g$, the velocity-increment at the surface of the earth due to the earth's attraction, (6) may be written

$$v = \frac{V - V_0}{g}. \qquad (6')$$

5. V_0 in the last two equations is the value of V where $v = 0$, or the value of V along the line of intersection of the disturbed and undisturbed surfaces. If we put

$$v_0 = V_0 \frac{r_0^2}{M}.$$

$$v + v_0 = V \frac{r_0^2}{M} = \frac{V}{g}. \qquad (7)$$

This equation represents the elevation of the disturbed surface above a spherical surface of equal potential, whose value is

$$\frac{M}{r_0 - v} = C_2,$$

since the difference between this and (5) gives (7).

The constant V_0 may be determined from the obvious condition that the disturbed and undisturbed surfaces contain equal volumes. It should be observed also in this connection, that V_0 is never of a lower order than V, and cannot therefore be neglected in comparison with V. This remark is important since some writers* have neglected V_0 and arrived at the erroneous equation

$$v = \frac{V}{g}$$

instead of (6′) or (7).

6. It is evident that the equations derived above, (5) to (7), will hold true if the disturbing mass m be a part of the earth's mass so long as the ratio m/M may be neglected in comparison with unity. Thus, m may represent the mass of a continent, the deficiency in mass of a lake or lake-basin, etc.

7. The next step requires the determination of the potential V of the attracting mass for any point of the disturbed surface, whether without or within the circle which we have assumed to define the boundary of the mass.

In order to derive an expression for V, let the rectangular and polar co-ordinates of any point of the disturbing mass be defined by the usual relations, viz.:—

$$x = r \cos\theta \cos\lambda,$$
$$y = r \cos\theta \sin\lambda,$$
$$z = r \sin\theta;$$

*Notably Archdeacon Pratt. See his *Figure of the Earth*, 4th edition, articles 200 and 213.

in which θ and λ correspond to polar distance and longitude respectively, the position of the origin being arbitrary. With reference to the same origin, let the co-ordinates of the attracted point on the sea-surface be

$$x' = r' \cos \theta' \cos \lambda',$$
$$y' = r' \cos \theta' \sin \lambda',$$
$$z' = r' \sin \theta'.$$

If D denote the distance between the attracting and attracted points and

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\lambda - \lambda'), \tag{8}$$

$$D^2 = r^2 + r'^2 - 2rr' \cos \psi = (r - r')^2 + 4rr' \sin^2 \tfrac{1}{2}\psi. \tag{9}$$

The volume-element of the attracting mass is

$$dx\, dy\, dz = r^2 dr \sin \theta\, d\theta\, d\lambda.$$

Hence if ρ denote the density, supposed uniform, of the attracting mass, a general expression for the required potential is

$$V = \rho \int\int\int \frac{r^2 dr \sin \theta\, d\theta\, d\lambda}{D}. \tag{10}$$

We must now evaluate this integral. Taking the centre of the sphere of reference as origin of co-ordinates, let

$$r = r_0 + u$$
and
$$r' = r_0 + v, \tag{11}$$

in which u and v are small quantities relative to r_0, v being the same of course as defined by (6). Premising what will be proved hereafter, namely, that quantities of the order u / r_0, v/r_0, and $(u - v)^2 / r_0^2$ may be neglected, equation (9) gives

$$D = 2r_0 \sin \tfrac{1}{2}\psi \tag{12}$$

From the first of (11) $\quad dr = du,$
and $\quad r^2 = r_0^2, \qquad (13)$

to terms of the order u / r.

Without loss of generality we may assume the line from which θ and θ' are reckoned to pass through the attracted point, and the plane from which λ and λ' are reckoned to pass through the attracted point and the centre of the attracting mass. In this case $\theta' = 0$, and $\lambda' = 0$, and (8) gives

$$\psi = \theta.$$

By means of this relation and the values in (12) and (13) equation (10) becomes

$$V = r_0 \rho \int\int\int du \cos \tfrac{1}{2}\theta \, d\theta \, d\lambda. \qquad (14)$$

Assuming for the present that the attracting mass is of uniform thickness, h, the limits of u will be 0 and h. Let the limits of θ, which are obviously functions of λ, be θ_1 and θ_2. The limits of λ are evidently equal in magnitude, but of opposite signs. Hence we have

$$V = 2r_0\rho \int_0^h \int_{\theta_1}^{\theta_2} \int_0^{\lambda} du \cos \tfrac{1}{2}\theta \, d\theta \, d\lambda = 4r_0 h\rho \int_0^{\lambda} (\sin \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1) \, d\lambda. \qquad (15)$$

To complete the evaluation of (15) it will be convenient to change variables. Consider the spherical triangles formed by the attracted and attracting points, the centre of the attracting mass, and the points in which the arc θ cuts the circle bounding the mass. Thus, in figures 1 and 2 let P be the attracting and A the attracted points, C the centre of the attracting mass and BDG the bounding circle. Then,

$$\theta = AP$$

and

$$\lambda = BAC.$$

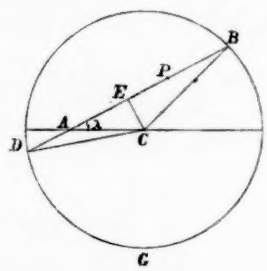


Fig. 1.

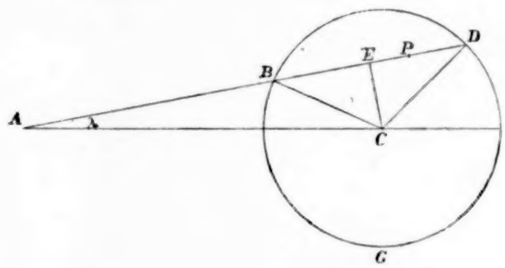


Fig. 2.

Draw CE perpendicular to AB and put

$$AC = \alpha, \qquad BC = \beta,$$
$$PE = s, \qquad BE = s_0,$$
$$CE = p, \qquad AE = q.$$

From either figure

$$\theta = q + s,$$
$$\theta_1 = q - s_0,$$
$$\theta_2 = q + s_0;$$

whence

$$\sin \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1 = 2 \cos \tfrac{1}{2}q \sin \tfrac{1}{2}s_0. \qquad (16)$$

The right-angled triangles of either figure give

$$\cos q = \frac{\cos \alpha}{\cos p},$$

$$\cos s_0 = \frac{\cos \beta}{\cos p}, \tag{17}$$

$$\sin p = \sin \alpha \sin \lambda.$$

The first two of (17) give

$$2 \cos^2 \tfrac{1}{2} q = 1 + \frac{\cos \alpha}{\cos p},$$

$$2 \cos^2 \tfrac{1}{2} s_0 = 1 - \frac{\cos \beta}{\cos p};$$

whence $$2 \cos \tfrac{1}{2} q \sin \tfrac{1}{2} s_0 = \frac{\sqrt{[(\cos p + \cos \alpha)(\cos p - \cos \beta)]}}{\cos p}. \tag{18}$$

From the last of (17)

$$d\lambda = \frac{\cos p \, dp}{\sqrt{(\cos^2 p - \cos^2 \alpha)}}. \tag{19}$$

Now the last of equations (17) and the diagrams show that the limits of p corresponding to the limits of λ are 0 and α, or 0 and β, according as the attracted point is within or without the circle bounding the attracting mass. Hence, if we denote the potentials in the two cases by V_1 and V_2 respectively, the equivalents in (15), (16), (18), and (19) give

$$V_1 = 4 r_0 h \rho \int_0^{\alpha} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \, . \, dp, \tag{20}$$

$$\alpha \leqq \beta$$

$$V_2 = 4 r_0 h \rho \int_0^{\beta} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \, . \, dp. \tag{21}$$

$$\alpha \geqq \beta$$

8. The integrals in these equations are in general elliptics of the third species. They may be evaluated by the usual processes applicable to elliptics, by mechanical quadrature, or by series. The integral in (20) presents some apparent difficulty, since the element-function is infinite at the upper limit except in the case $\alpha = \beta$. Again in case $\alpha = 0$, the integral assumes this anomalous form

$$\int_0^0 \sqrt{\left(\frac{1 - \cos \beta}{1 - 1}\right)} \, . \, dp,$$

the value of which is $\pi \sin \frac{1}{2}\beta$, as may be easily verified by means of (15), (16), and (18). These peculiar features may be removed by the following transformation, which secures the same constant limits for both (20) and (21).

For brevity put

$$I_1 = \int_0^{\alpha} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \cdot dp, \tag{22}$$

$$I_2 = \int_0^{\beta} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \cdot dp. \tag{23}$$

Then, observing that $\dfrac{\cos p - \cos \beta}{\cos p - \cos \alpha} = \dfrac{\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}p}{\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}p}$,

put in I_1 $\qquad \sin \frac{1}{2}p = \sin \frac{1}{2}\alpha \sin \gamma_1$,

and in I_2 $\qquad \sin \frac{1}{2}p = \sin \frac{1}{2}\beta \sin \gamma_2$.

These give

$$dp = \frac{2 \sin \frac{1}{2}\alpha \cos \gamma_1 \, d\gamma_1}{\sqrt{(1 - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1)}},$$

$$dp = \frac{2 \sin \frac{1}{2}\beta \cos \gamma_2 \, d\gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}},$$

and the limits for both γ_1 and γ_2 are 0 and $\frac{1}{2}\pi$. Therefore

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{2 \sin \frac{1}{2}\beta \sqrt{\left(1 - \frac{\sin^2 \frac{1}{2}\alpha}{\sin^2 \frac{1}{2}\beta} \sin^2 \gamma_1\right)} \, d\gamma_1}{\sqrt{(1 - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1)}}, \tag{24}$$

$$\alpha \leqq \beta$$

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{2 \sin^2 \frac{1}{2}\beta \cos^2 \gamma_2 \, d\gamma_2}{\sin \frac{1}{2}\alpha \sqrt{\left(1 - \frac{\sin^2 \frac{1}{2}\beta}{\sin^2 \frac{1}{2}\alpha} \sin^2 \gamma_2\right)} \cdot \sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}}. \tag{25}$$

$$\alpha \geqq \beta$$

[TO BE CONTINUED.]

ON THE USE OF SOMOFF'S THEOREM FOR THE EVALUATION OF THE ELLIPTIC INTEGRAL OF THE THIRD SPECIES.

By Mr. Chas. H. Kummell, Washington, D. C.

[continued from vol. ii, page 77.]

For the convenient computation of the elements of the next higher step, I propose the following algorithm. Starting with $a = 1$, $b = \beta$, $c = \gamma$, $a\Delta\varphi$, $c\cos\varphi$, $a\Delta\mu$, $c\cos\mu$, as data, compute as follows: —

$$\begin{aligned} a' &= \tfrac{1}{2}(a + c), & b' &= \tfrac{1}{2}(a - c), \quad c' = \sqrt{(ac)}\,; \\ a'\Delta\varphi' &= \tfrac{1}{2}(a\Delta\varphi + c\cos\varphi), & c'\cos\varphi' &= \sqrt{(a'\Delta\varphi' + b')}\,.\sqrt{(a'\Delta\varphi' - b')}, \\ a'\Delta\mu' &= \tfrac{1}{2}(a\Delta\mu + c\cos\mu), & c'\cos\mu' &= \sqrt{(a'\Delta\mu' + b')}\,.\sqrt{(a'\Delta\mu' - b')}, \\ a'\Delta\mu^{\backprime} &= \tfrac{1}{2}(a\Delta\mu - c\cos\mu), & c'\cos\mu^{\backprime} &= \sqrt{(a'\Delta\mu^{\backprime} + b')}\,.\sqrt{(a'\Delta\mu^{\backprime} - b')}. \end{aligned} \tag{18}$$

We thus obtain the new elements of the next higher step in the modular scale, viz. $\gamma' = c'/a'$, φ', μ', $\mu^{\backprime}$, from which (16) can be formed. Computing in the same manner a'', b'', c'' from a', b', c'; $a''\Delta\varphi''$ and $c''\cos\varphi''$ from $a'\Delta\varphi'$ and $c'\cos\varphi'$; $a''\Delta\mu''$ and $c''\cos\mu''$, as well as $a''\Delta\mu'^{\backprime}$ and $c''\cos\mu'^{\backprime}$, from $a'\Delta\mu'$ and $c'\cos\mu'$; $a''\Delta\mu^{\backprime\prime}$ and $c''\cos\mu^{\backprime\prime}$, as well as $a''\Delta\mu^{\backprime\backprime}$ and $c''\cos\mu^{\backprime\backprime}$, from $a'\Delta\mu^{\backprime}$ and $c'\cos\mu^{\backprime}$, we obtain the elements of the equation

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Pi(\varphi''_{\gamma''}, \mu''_{\gamma''}) - \Pi(\varphi''_{\gamma''}, \mu'^{\backprime}_{\gamma''}) - \Pi(\varphi''_{\gamma''}, \mu^{\backprime\prime}_{\gamma''}) + \Pi(\varphi''_{\gamma''}, \mu^{\backprime\backprime}_{\gamma''}). \tag{16''}$$

Continuing this process until $a^{(n)} = c^{(n)} =$ arithmetico-geometric mean of a and c, and also $a^{(n)}\Delta\varphi^{(n)} = c^{(n)}\cos\varphi^{(n)}$, $a^{(n)}\Delta\mu^{(n-1)'} = c^{(n)}\cos\mu^{(n-1)'}$, but $\Delta\mu^{(n-1)'} = \cos\mu^{(n-1)\backprime} = 0$, then we have finally

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Sigma\left[(-)^s\, \Pi\left(\varphi^{(n)}_{\gamma^{(n)}}, \mu^{(n-1)'}_{\gamma^{(n)}}\right)\right] = \Sigma\left[(-)^s\, \Pi\left(\varphi^{(n)}_1, \mu^{(n-1)'}_1\right)\right], \tag{16^{(n)}}$$

where s denotes the number of grave accents affixed to μ.

The integral is now expressed in 2^n terms of the same form; viz.: —

$$\Pi\left(\varphi^{(n)}_1, \mu^{(n-1)'}_1\right) = \int_0^{\phi^{(n)}} \sin\mu^{(n-1)'}\cos^2\mu^{(m-1)'}\cdot\frac{d\varphi}{\cos\varphi}\cdot\frac{\sin^2\varphi}{1 - \sin^2\mu^{(n-1)'}\sin^2\varphi}$$

$$= \tfrac{1}{2}\sin\mu^{(n-1)'}\, l\,\frac{1 + \sin\varphi^{(n)}}{1 - \sin\varphi^{(n)}} - \tfrac{1}{2}\, l\,\frac{1 + \sin\mu^{(n-1)'}\sin\varphi^{(n)}}{1 - \sin\mu^{(n-1)'}\sin\varphi^{(n)}}. \tag{19}$$

This is the standard analytical form. To adapt it to the use of logarithmic tables, place

$$\cos\lambda^{(n-1)'} = \sin\mu^{(n-1)'}\sin\varphi^{(n)}\,; \tag{20}$$

then, since $$\varphi_\gamma = \frac{1}{2a^{(n)}}\, l\,\frac{1 + \sin\varphi^{(n)}}{1 - \sin\varphi^{(n)}} = \frac{1}{c^{(n)}}\, l\tan\tfrac{1}{2}(\perp + \varphi^{(n)}), \tag{21}$$

we have

$$\Pi\,(\varphi_1^{(n)},\ \mu_1^{(n-1)\prime}) \log e = \sin \mu^{(n-1)\prime} \log \tan \tfrac{1}{2}\,(\,\lrcorner + \varphi^{(n)}) - \log \cot \tfrac{1}{2}\lambda^{(n-1)\prime}$$
$$= c^{(n)} \sin \mu^{(n-1)\prime}\, \varphi_\gamma \log e - \log \cot \tfrac{1}{2}\lambda^{(n-1)\prime}. \qquad (19')$$

Computing this quantity for each limit, and combining according to $(16^{(n)})$, we obtain $\Pi\,(\varphi_\gamma,\ \mu_\gamma) \log e$, whence $\Pi\,(\varphi_\gamma,\ \mu_\gamma)$.

This method is not applicable to the cyclometric integral $\Pi\,(\varphi_\gamma,\ \nu_\beta i)$, as I had at first thought, although the algorithm (18) is applicable to $a\,\Delta\,(\nu_\beta i)_{-\gamma} = a\,\Delta\nu$ sec ν, $c \cos\,(\nu_\beta i)_{-\gamma} = c \sec \nu$ in a straight line, because these and $a'\Delta\,(\nu'_{\beta'} i)_{-\gamma'}$, $c' \cos\,(\nu'_{\beta'} i)_{-\gamma'}$ and $a''\Delta\,(\nu''_{\beta''} i)_{-\gamma''}$, $c'' \cos\,(\nu''_{\beta''} i)_{-\gamma''}$, etc., are all real. But, although $a'\Delta\,(\nu'_{\beta'} i)_{-\gamma'}$ is also real, $c' \cos\,(\nu'_{\beta'} i)_{-\gamma'} = \sqrt{[a'^2\Delta^2(\nu'_{\beta'} i)_{-\gamma'} - b'^2]}$ is not, because $a'\Delta\,(\nu'_{\beta'} i)_{-\gamma'} < b'$. This makes the above method, which I shall distinguish as the first method, impracticable for the cyclometric integral, at least for the ascending scale.

There is a slight defect in this method in case φ is small; for the algorithm (18) ends with $a^{(n)}\,\Delta\varphi^{(n)} = c^{(n)} \cos \varphi^{(n)}$, or $\Delta\varphi^{(n)} = \cos \varphi^{(n)}$, which is near unity if φ, and hence $\varphi^{(n)}$, are small. The determination of $\varphi^{(n)}$ from the cosine is in that case not precise.

Now, since
$$\sin \varphi = (1 + \beta')\frac{\sin \varphi' \cos \varphi'}{\Delta\varphi'}$$
$$= \frac{a \sin \varphi' \cos \varphi'}{a'\Delta\varphi'},$$

we have
$$c \sin \varphi = \frac{c' \sin \varphi' .\ c' \cos \varphi'}{a'\Delta\varphi'};$$

$$\therefore\ c' \sin \varphi' = \frac{a'\Delta\varphi'}{c' \cos \varphi'}.\ c \sin \varphi. \qquad (22)$$

Adding this easy computation for each step of the modular scale, we have finally both $c^{(n)} \sin \varphi^{(n)}$ and $c^{(n)} \cos \varphi^{(n)}$ from the algorithm, and $\varphi^{(n)}$ can be determined from its sine, cosine, or tangent. Moreover, since (20) requires $\sin \varphi^{(n)}$ and $\sin \mu^{(n)}$, this results directly.

While this controlling computation is not essential to this method, it is to the second method, which is based on form (17) of Somoff's theorem. We have, at the second step of the scale,

$$\Pi\,(\varphi_\gamma,\ \mu_\gamma) = 2^2\ \Pi\,(\varphi''_{\gamma''},\ \mu''_{\gamma''}) - 2\left(\gamma' \sin \mu' \varphi'_{\gamma'} - \tfrac{1}{2} l\,\frac{1 + \gamma' \sin \mu' \sin \varphi'}{1 - \gamma' \sin \mu' \sin \varphi'}\right)$$
$$- \gamma \sin \mu\, \varphi_\gamma + \tfrac{1}{2} l\,\frac{1 + \gamma \sin \mu \sin \varphi}{1 - \gamma \sin \mu \sin \varphi}$$

$$= 2^2 \Pi(\varphi''_{\gamma''}, \mu''_{\gamma''}) - (2c' \sin\mu' + c\sin\mu)\,\varphi_\gamma + l\cot^2\tfrac{1}{2}\lambda' \cot\tfrac{1}{2}\lambda, \tag{23}$$

where $$\cos\lambda^{(0)} = \gamma\sin\mu\sin\varphi = \frac{c\sin\mu \,.\, c\sin\varphi}{a\,.\,c} \tag{24}$$

$$\cos\lambda' = \gamma'\sin\mu'\sin\varphi' = \frac{c'\sin\mu' .\, c'\sin\varphi'}{a'.\,c'}; \tag{24'}$$

and finally, at the limit,

$$\Pi(\varphi_\gamma, \mu_\gamma) = 2^n \Pi(\varphi_1^{(n)}, \mu_1^{(n)}) - \varphi_\gamma(2^{n-1}c^{(n-1)}\sin\mu^{(n-1)} + \ldots + 2c'\sin\mu' + c\sin\mu)$$
$$+ \quad 2^{n-1} l\cot\tfrac{1}{2}\lambda^{(n-1)} + \ldots + 2l\cot\tfrac{1}{2}\lambda' + l\cot\tfrac{1}{2}\lambda^{(0)}. \tag{25}$$

The first term is, of course, computed by (19′). If, however, the control (22) is used, the computation of the limiting step may be omitted; for, since then

$$a^{(n)}\,\Delta\varphi^{(n)} = c^{(n)}\cos\varphi^{(n)},$$

we have by (22) $$c^{(n)}\sin\varphi^{(n)} = c^{(n-1)}\sin\varphi^{(n-1)},$$

$$c^{(n)}\sin\mu^{(n)} = c^{(n-1)}\sin\mu^{(n-1)};$$

therefore $$\cos\lambda^{(n)} = \cos\lambda^{(n-1)},$$

and $$\Pi(\varphi_1^{(n)}, \mu_1^{(n)}) = c^{(n)}\sin\mu^{(n)}\,\varphi_\gamma - l\cot\tfrac{1}{2}\lambda^{(n)}$$

$$= c^{(n-1)}\sin\mu^{(n-1)}\,\varphi_\gamma - l\cot\tfrac{1}{2}\lambda^{(n-1)}.$$

Regarding this, we have, instead of (25),

$$\Pi(\varphi_\gamma, \mu_\gamma)$$
$$= \varphi_\gamma(2^{n-1}c^{(n-1)}\sin\mu^{(n-1)} - 2^{n-2}c^{(n-2)}\sin\mu^{(n-2)} - \ldots - 2c'\sin\mu' - c\sin\mu)$$
$$- 2^{(n-1)} l\cot\tfrac{1}{2}\lambda^{(n-1)} + 2^{(n-2)} l\cot\tfrac{1}{2}\lambda^{(n-2)} + \ldots + 2\,l\cot\tfrac{1}{2}\lambda' + l\cot\tfrac{1}{2}\lambda. \tag{25'}$$

Now, with a slight modification, this method may be used for the cyclometric integral. Placing, for the sake of uniformity,

$$\mu_\gamma = \nu_\beta i, \tag{26}$$

then the algorithm for parameter starts with

$$a\,\Delta\mu = a\,\Delta(\nu_\beta i)_{-\gamma} = a\,\Delta\nu \sec\nu \tag{27}$$

and $$c\cos\mu = c\cos(\nu_\beta i)_{-\gamma} = c\sec\nu, \tag{28}$$

and is real in a straight line. The control (22) becomes

$$c'\sin\mu' = c'\sin(\nu'_{\beta'} i)_{-\gamma'} = \frac{a'\Delta\mu'}{c'\cos\mu'} \cdot c\sin\mu = ic'\tan\nu',$$

or $$c'\tan\nu' = \frac{a'\Delta\mu'}{c'\cos\mu'} . c\tan\nu. \tag{29}$$

The formula (24) cannot be used, since λ is not real. Placing, instead,

$$\tan x^{(0)} = \frac{c \tan \nu \, . \, c \sin \varphi}{a \, . \, c}, \tag{30}$$

we have

$$\tfrac{1}{2} l \frac{1 + \gamma \sin \mu \sin \varphi}{1 - \gamma \sin \mu \sin \varphi} = \tfrac{1}{2} l \frac{1 + \gamma i \tan \nu \sin \varphi}{1 - \gamma i \tan \nu \sin \varphi}$$
$$= \tfrac{1}{2} l \frac{1 + i \tan x^{(0)}}{1 - i \tan x^{(0)}}$$
$$= i x^{(0)}, \tag{31}$$

and therefore

$$\Pi(\varphi_\gamma, \nu_\beta i)$$
$$= \varphi_\gamma i (2^{n-1} c^{(n-1)} \tan \nu^{(n-1)} - 2^{n-2} c^{(n-2)} \tan \nu^{(n-2)} - \ldots - 2c' \tan \nu' - c \tan \nu)$$
$$- i (2^{n-1} x^{(n-1)} - 2^{n-2} x^{(n-2)} - \ldots - 2x' - x^{(0)}). \tag{32}$$

There is, however, a modification of these methods required, if the modulus γ is very small; for, although even then the method necessarily converges, it will do so after a greater number of steps have been ascended, which makes the computation both tedious and less precise. If γ is small then β is near unity. The required modification is therefore obtained by reducing the integral to the complementary modulus by Jacobi's imaginary transformation. We have thus

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Pi(\psi_\beta i, \nu_\beta i)$$
$$= \int_0^{\psi_\beta i} \sin(\nu_\beta i)_{-\gamma} \cos(\nu_\beta i)_{-\gamma} \Delta(\nu_\beta i)_{-\gamma} \cdot \frac{d\varphi}{\Delta\varphi} \cdot \frac{\gamma^2 \sin \varphi}{1 - \gamma^2 \sin^2(\nu_\beta i)_{-\gamma} \sin^2 \varphi}$$
$$= \int_0^{\psi} i \tan \nu \frac{\Delta(\nu_\beta)_{-\beta}}{\cos^2 \nu} \, . \, i \frac{d\psi}{\Delta(\psi_\beta)_{-\beta}} \cdot \frac{-\gamma^2 \tan^2 \psi}{1 - \gamma^2 \tan^2 \nu \tan^2 \psi}$$
$$= \int_0^{\psi} \tan \nu \frac{\Delta\nu}{\cos^2 \nu} \cdot \frac{d\psi}{\Delta\psi} \cdot \frac{\gamma^2 \sin^2 \psi}{1 - \dfrac{\Delta^2 \nu}{\cos^2 \nu} \sin^2 \psi}. \tag{33}$$

Since

$$\sin(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta} = \frac{\Delta\nu}{\beta \cos \nu},$$
$$\cos(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta} = -\frac{i\gamma}{\beta \cos \nu},$$
$$\Delta(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta} = i\gamma \tan \nu;$$

we have

$$\beta^2 \sin(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta} \cos(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta} \Delta(\text{⅃}_\beta + \text{⅃}_\gamma i + \nu_\beta)_{-\beta}$$

$$= \gamma^2 \tan\nu \frac{\Delta\nu}{\cos^2\nu},$$

and therefore

$$\Pi(\varphi_\gamma, \mu_\gamma)$$

$$= \int_0^\psi \left\{ \begin{array}{c} \sin(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \cos(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \Delta(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \\ \times \dfrac{d\psi}{\Delta\psi} \cdot \dfrac{\beta^2 \sin^2\psi}{1 - \beta^2 \sin^2(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \sin^2\psi} \end{array} \right\}$$

$$= \Pi(\psi_\beta, \lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)$$

$$= \Pi(\psi_\beta, \nu_\beta) - \psi_\beta \tan\nu\, \Delta\nu + \tfrac{1}{2}\, l \frac{\cos(\nu_\beta - \psi_\beta)_{-\beta}}{\cos(\nu_\beta + \psi_\beta)_{-\beta}}, \text{ by (9),}$$

$$= \Pi(\psi_\beta, \nu_\beta) - \frac{\varphi_\gamma}{i} \cdot \frac{\tan\mu\,\Delta\mu}{i} + \tfrac{1}{2}\, l \frac{1 + \frac{1}{i}\tan\mu\,\Delta\mu \,.\, \frac{1}{i}\tan\varphi\,\Delta\varphi}{1 - \frac{1}{i}\tan\mu\,\Delta\mu \,.\, \frac{1}{i}\tan\varphi\,\Delta\varphi}$$

$$= \Pi(\psi_\beta, \nu_\beta) + \varphi_\gamma \tan\mu\,\Delta\mu - \tfrac{1}{2}\, l \frac{\cos(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\cos(\mu_\gamma + \varphi_\gamma)_{-\gamma}} \tag{34}$$

$$= \Pi(\psi_\beta, \nu_\beta) + \varphi_\gamma \tan\mu\,\Delta\mu - l \cot\tfrac{1}{2}\omega, \text{ if } \cos\omega = \tan\mu\,\Delta\mu \tan\varphi\,\Delta\varphi. \tag{35}$$

The term $\Pi(\psi_\beta, \nu_\beta)$ is treated like a logarithmic integral to the modulus β. We start the algorithm (18) and control (22) with the data

$$a = 1, \qquad c = \gamma, \qquad b = \beta;$$

$$a\,\Delta\psi = a\,\Delta\varphi \sec\varphi,\ b\cos\psi = b\sec\varphi,\ b\sin\psi = \frac{b}{i}\tan\varphi;$$

$$a\,\Delta\nu = a\,\Delta\mu \sec\mu,\ b\cos\nu = b\sec\mu,\ b\sin\nu = \frac{b}{i}\tan\mu; \tag{36}$$

and computing the next higher step of β, being the next lower step to γ, the quantities

$$a_{,} = \tfrac{1}{2}(a + b),\ c_{,} = \tfrac{1}{2}(a - b),\ b_{,} = \sqrt{(ab)};$$

$$a_{,}\,\Delta\psi_{,} = \tfrac{1}{2}(a\,\Delta\psi + b\cos\psi) = a\,\Delta\varphi_{,} \sec\varphi_{,},$$

$$b_{,}\cos\psi_{,} = \sqrt{(a_{,}^2\Delta^2\psi_{,} - c_{,}^2)} = b_{,}\sec\varphi_{,},$$

$$b_{,}\sin\psi_{,} = \frac{a_{,}\,\Delta\psi_{,}}{b_{,}\cos\psi_{,}} \,.\, b\sin\psi = \frac{b_{,}}{i}\tan\varphi_{,};$$

$$a_{,}\,\Delta\nu_{,} = \tfrac{1}{2}(a\,\Delta\nu + b\cos\nu) = a\,\Delta\mu_{,} \sec\mu_{,},$$

$$b_{,}\cos\nu_{,} = \sqrt{(a_{,}^2\Delta^2\nu_{,} - c_{,}^2)} = b_{,}\sec\mu_{,},$$

$$b_{,}\sin\nu_{,} = \frac{a_{,}\,\Delta\nu_{,}}{b_{,}\cos\nu_{,}} \,.\, b\sin\nu = \frac{b_{,}}{i}\tan\mu_{,}. \tag{36'}$$

Continuing in this manner until $a_{(n)} = b_{(n)}$ = arithmetico-geometric mean of a and b, we have also $a_{(n)}\Delta\psi_{(n)} = b_{(n)}\cos\psi_{(n)}$ or $\Delta\varphi_{(n)}\sec\varphi_{(n)} = \sec\varphi_{(n)}$, and simi-

larly for μ. By (25′) we have now, going back to its more fundamental form,

$$\Pi(\psi_\beta, \nu_\beta)$$

$$= \psi_\beta (2^{n-1} b_{(n-1)} \sin \nu_{(n-1)} - 2^{n-2} b_{(n-2)} \sin \nu_{(n-2)} - \ldots - 2b_, \sin \nu_, - b \sin \nu)$$

$$- 2^{n-2} l \frac{1 + \beta_{(n-1)} \sin \nu_{(n-1)} \sin \psi_{(n-1)}}{1 - \beta_{(n-1)} \sin \nu_{(n-1)} \sin \psi_{(n-1)}}$$

$$+ 2^{n-3} l \frac{1 + \beta_{(n-2)} \sin \nu_{(n-2)} \sin \psi_{(n-2)}}{1 - \beta_{(n-2)} \sin \nu_{(n-2)} \sin \psi_{(n-2)}}$$

$$+ \ldots$$

$$+ l \frac{1 + \beta_, \sin \nu_, \sin \psi_,}{1 - \beta_, \sin \nu_, \sin \psi_,} + \tfrac{1}{2} l \frac{1 + \beta \sin \nu \sin \psi}{1 - \beta \sin \nu \sin \psi}$$

$$= -\varphi_\gamma (2^{n-1} b_{(n-1)} \tan \mu_{(n-1)} - 2^{n-2} b_{(n-2)} \tan \mu_{(n-2)} - \ldots - 2b_, \tan \mu_, - b \tan \mu)$$

$$+ 2^{n-1} l \cot \tfrac{1}{2}\lambda_{(n-1)} - 2^{n-2} l \cot \tfrac{1}{2}\lambda_{(n-2)} - \ldots - 2l \cot \tfrac{1}{2}\lambda_, - l \cot \tfrac{1}{2}\lambda_{(0)}, \quad (37)$$

where we have placed

$$\cos \lambda_{(0)} = \beta \tan \mu \tan \varphi = \frac{b \tan \mu \,.\, b \tan \varphi}{a \,.\, b},$$

$$\cos \lambda_, = \beta_, \tan \mu_, \tan \varphi_, = \frac{b_, \tan \mu_, \,.\, b_, \tan \varphi_,}{a_, \,.\, b_,}, \text{ etc.} \quad (38)$$

We have also here

$$\varphi_\gamma = i\psi_\beta = \frac{i}{2b_{(n)}} l \frac{1 + \sin \psi_{(n)}}{1 - \sin \psi_{(n)}}$$

$$= \frac{\varphi_{(n)}}{b_{(n)}}, \quad (39)$$

and we recognize in this the elegant formula of Gauss for the integral of the first species, and $\varphi_{(n)}$ is the argument of Jacobi's ϑ functions, usually denoted by x.

Treating the cyclometric integral in a similar manner, we have

$$\Pi(\varphi_\gamma, \nu_\beta i) = \Pi(\psi_\beta i, -\mu_\gamma)$$

$$= \Pi(\psi_\beta, J_\beta + J_\gamma i - \mu_\gamma i)$$

$$= \Pi(\psi_\beta, J_\beta + J_\gamma i - \nu_\beta)$$

$$= \Pi(\psi_\beta, -\nu_\beta) + \psi_\beta \tan \nu \Delta\nu - \tfrac{1}{2} l \frac{\cos(\nu_\beta - \psi_\beta)_{-\beta}}{\cos(\nu_\beta + \psi_\beta)_{-\beta}}$$

$$= -\Pi(\psi_\beta, \nu_\beta) - i[\varphi_\gamma \tan \nu \Delta\nu - \text{arc tan}(\tan \nu \Delta\nu \tan \varphi \Delta\varphi)]$$

$$= -\Pi(\psi_\beta, \nu_\beta) - i[\varphi_\gamma \tan \nu \Delta\nu - o), \text{ if } \tan o = \tan \nu \Delta\nu \tan \varphi \Delta\varphi. \quad (40)$$

For the first term the algorithm starts with the data

$$a = 1, \qquad c = \gamma, \qquad b = \beta;$$

$$a\,\Delta\psi = a\,\Delta\varphi \sec\varphi, \quad b\cos\psi = b\sec\varphi, \quad b\sin\psi = \frac{b}{i}\tan\varphi;$$

$$a\,\Delta\nu, \qquad b\cos\nu, \qquad b\sin\nu; \tag{41}$$

and, since ν is real, we can pursue the algorithm for ν also sideways, by computing also $a_{\prime}\,\Delta\nu_{\prime} = \frac{1}{2}(a\,\Delta\nu + b\cos\nu)$ and following up each branch to its limit. We have then, according to the first method,

$$\Pi(\psi_\beta, \nu_\beta) = \Sigma[(-)^{\nu}\,\Pi([\psi_{(n)}]_1, [\nu_{(n-1)\prime}]_1)], \tag{42}$$

where

$$\Pi([\psi_{(n)}]_1, [\nu_{(n-1)}]_1) = b_{(n)}\sin\nu_{(n-1)\prime}\,\psi_\beta - \tfrac{1}{2}l\,\frac{1 + \sin\nu_{(n-1)\prime}\sin\psi_{(n)}}{1 - \sin\nu_{(n-1)\prime}\sin\psi_{(n)}}$$

$$= -\,b_{(n)}\,i\sin\nu_{(n-1)\prime}\,\varphi_\gamma + i\arctan(\sin\nu_{(n-1)\prime}\tan\varphi_{(n)})$$

$$= -\,i(b_{(n)}\sin\nu_{(n-1)\prime}\,\varphi_\gamma - \varkappa_{(n-1)\prime}),$$

if

$$\tan\varkappa_{(n-1)\prime} = \sin\nu_{(n-1)\prime}\tan\varphi_{(n)}. \tag{43}$$

By the second method we have

$$\begin{aligned}
\Pi(\psi_\beta, \nu_\beta)
&= \psi_\beta(2^{n-1}b_{(n-1)}\sin\nu_{(n-1)} - 2^{n-2}b_{(n-2)}\sin\nu_{(n-2)} - \ldots - 2b_{\prime}\sin\nu_{\prime} - b\sin\nu)\\
&\quad - 2^{n-2}\,l\,\frac{1 + \beta_{(n-1)}\sin\nu_{(n-1)}\sin\psi_{(n-1)}}{1 - \beta_{(n-1)}\sin\nu_{(n-1)}\sin\psi_{(n-1)}}\\
&\quad + 2^{n-3}\,l\,\frac{1 + \beta_{(n-2)}\sin\nu_{(n-2)}\sin\psi_{(n-2)}}{1 - \beta_{(n-2)}\sin\nu_{(n-2)}\sin\psi_{(n-2)}}\\
&\quad + \ldots\\
&\quad + l\,\frac{1 + \beta_{\prime}\sin\nu_{\prime}\sin\psi_{\prime}}{1 - \beta_{\prime}\sin\nu_{\prime}\sin\psi_{\prime}} + \tfrac{1}{2}l\,\frac{1 + \beta\sin\nu\sin\psi}{1 - \beta\sin\nu\sin\psi}\\
&= -\,\varphi_\gamma\, i(2^{n-1}b_{(n-1)}\sin\nu_{(n-1)} - 2^{n-2}b_{(n-2)}\sin\nu_{(n-2)} - \ldots - 2b_{\prime}\sin\nu_{\prime} - b\sin\nu)\\
&\quad + i(2^{n-1}\varkappa_{(n-1)} - 2^{n-2}\varkappa_{(n-2)} - \ldots - 2\varkappa_{\prime} - \varkappa_{(0)}),
\end{aligned} \tag{44}$$

where

$$\tan\varkappa_{(s)} = \beta_{(s)}\sin\nu_{(s)}\tan\varphi_{(s)} = \frac{b_{(s)}\sin\nu_{(s)}\,.\,b_{(s)}\tan\varphi_{(s)}}{a_{(s)}\,.\,b_{(s)}}. \tag{45}$$

I have thus shown that the algorithm which forms the main part of the computation is applicable in all cases of the integral of the third species which need to be considered. The practical use of the methods will be shown in a future article.

THE FIGURE OF THE EARTH AND THE MOTION OF THE MOON.

By Prof. Asaph Hall, Washington, D. C.

Having recently examined the history of this question, and finding no connected account of the discovery of the inequalities in the motion of the moon depending on the figure of the earth, the following notes may be of some interest: —

1. The inequality of the longitude of the moon arising from the figure of the earth was found first by Tobias Mayer of Göttingen about 1753. It appears that Mayer discovered this term from the discussion of his own observations of the moon. He made the value of the co-efficient $4''$; but as he gave no theoretical reason for the inequality, and as the co-efficient was small with respect to the accuracy of the observations of those times, most of the mathematicians neglected it. But the lunar tables constructed by Mayer were compared with the Greenwich observations by Bradley, 1756 to 1760, who found these tables far superior to any others that had been made. Mayer died in 1762 at the age of thirty-nine, and the English government gave to his widow the sum of three thousand pounds sterling for his lunar tables. It is remarkable, I think, that Mayer was able to detect so small an inequality from his observations of the moon, which could not have been very extensive; but he seems to have been an astronomer who combined sound mathematical knowledge with great skill and judgment in the discussion of astronomical and physical questions.

2. In 1780, Mayer's tables were compared with observations by Charles Mason, who had been an assistant of Bradley, and a new edition was published in 1787. The co-efficients were corrected, and that for the inequality in the longitude which depends on the figure of the earth was found to be $7''.7$.

3. About 1800 Bürg of Vienna found by comparing a large number of observations the value of this co-efficient to be $6''.8$. Up to this time, no direct theoretical investigation of this question seems to have been made, although, as will be noticed, the period includes the time when Legendre and Laplace were making their celebrated researches on the attraction of ellipsoids.

a. The first theoretical investigation of this question that I have found was by Lagrange in 1773 in his prize memoir of the Paris Academy of 1774. The memoir of Lagrange is on the secular equation of the moon, and he examines the action of the figure of the earth in this equation, and shows that there is no term arising from this source that would produce a secular effect. The investigation of Lagrange is an elaborate one, and covers nearly fifty pages in the sixth volume of his works (edition of Serret). As the purpose was to find only secular terms, the periodical ones were not computed. Lagrange had already con

sidered a similar question in his prize memoir on the inequalities of the satellites of Jupiter, 1766, where he gives the expressions of the forces arising from the figure of Jupiter.

β. In 1800, at the request of Bürg, as it appears, Laplace first computed the co-efficients of the principal inequalities produced by the figure of the earth in the longitude and latitude of the moon. The memoir was read "*le 26 prairial an 8*," and is printed in the third volume of the memoirs of the Institute, p. 198. It covers only nine pages, and has been reproduced in the *Mécanique Céleste*, Vol. III. With an assumed figure of the earth, Laplace finds the co-efficient in longitude 5".6, and in latitude — 6".5. He points out that in case of the homogeneity of the earth these co-efficients would be much larger, and as this is contradicted by observation, the earth cannot be homogeneous. It appears from this memoir that Laplace had previously made an erroneous calculation, in which he found the co-efficients of these inequalities less than 2".

γ. It may be worth while to mention in this connection two of the recent determinations of these co-efficients. In his theory of the moon, 1864, Hansen with a new value for the figure of the earth finds the value of the co-efficients 7".40 and — 8".38. Mr. G. W. Hill, in a recent paper from the office of the American Ephemeris and Nautical Almanac, 1884, has made an elaborate investigation of this question, and finds for the principal terms 7".67 and — 8".73.

These inequalities are interesting, since, combined with the figure of the earth determined from geodetic measurements, they give us some idea of the constitution of our planet. Speaking of these terms, Laplace says: "Les deux inégalités précédentes méritent toute l'attention des observateurs; elles ont sur les mesures géodésiques l'avantage de donner l'aplatissement de la terre d'une manière moins dépendante des irregularités de sa figure."

There is another interesting inequality in the moon's motion, the parallactic equation, which depends on the angular distance of the sun and moon, and which furnishes a means of determining the sun's distance from the earth, or the solar parallax. Tobias Mayer seems to have been the first astronomer who used this inequality for determining the sun's distance. He was also one of the first astronomers to employ equations of condition in the discussion of observations.

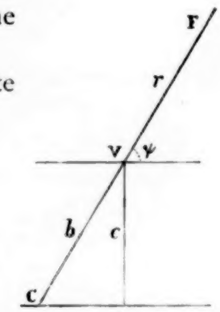
1886. July 5.

SOLUTIONS OF EXERCISES.

40

THE VERTEX of an hyperbola and one asymptote is fixed. Find the locus of the focus.

SOLUTION.



Let r, ψ be the polar co-ordinates of the focus, referred to the vertex and to a line through the vertex parallel to the asymptote,

c, b, the distances of the vertex from the asymptote and from the centre,

$e = \sec\psi$, the eccentricity.

Then, as may easily be seen,

$$b \sin\psi = c,$$
$$r = b(\sec\psi - 1);$$
$$\therefore\ r\sin\psi = c(\sec\psi - 1),$$

or
$$r = c\tan\tfrac{1}{2}\psi \,.\, \sec\psi,$$

the equation to the locus sought. Expressed in rectangular co-ordinates, this becomes, after reduction,

$$x^2(y + 2c) = c^2y,$$

a cubic, symmetrical with the axis of y. The curve has three branches A, B, C and three asymptotes a, b, c, of which a is parallel to the axis of x, and b and c are parallel to the axis of y; a is common to B and C, b to A and C, c to A and B. [*Ormond Stone.*]

50

ABC is an equilateral triangle inscribed in a circle. From any point P within the triangle BAC, straight lines PA, PB, PC are drawn. Show, *geometrically*, that $PB + PC - PA$ is a minimum when P is on the circumference of the circle about ABC. [*R. D. Bohannan.*]

SOLUTION.

Let P_1, P_2 be two points near the point P, and so selected that $CP_1 = CP_2$. From the nature of minimum values we have

$$P_1B + P_1C - P_1A = P_2B + P_2C - P_2A;$$
$$\therefore\ P_1B - P_1A = P_2B - P_2A;$$

$\therefore$ P_1, P_2 are on an hyperbola having A, B for foci. Letting P_1, P_2 come indefinitely near P, P_1P_2 is the tangent to the hyperbola; but, since $CP_1 = CP_2$,

P_1P_2 is perpendicular to CP, PC is normal to the hyperbola, and bisects the angle between PB and PA prolonged.

In like manner, selecting P_1, P_2 so that $P_1A = P_2A$, we have

$$P_1B + P_1C = P_2B + P_2C;$$

$\therefore$ P is on an ellipse whose foci are B, C. P_1P_2 is tangent to this ellipse; $\therefore$ P_1A is normal; $\therefore$ PA bisects the angle BPC.

Thus, calling PD the prolongation of CP, and PE the prolongation of AP, we have

$$\angle APB = \angle APC = \angle DPE = \angle DPB;$$
$$\therefore \angle APB = 60^\circ;$$

$\therefore$ P moves on a circle about ABC. [*R. D. Bohannan.*]

72

Circles of given radius are drawn through the focus of a fixed parabola, cutting the curve in four points. Show that the products of the focal radii to these points are all equal. [*W. M. Thornton.*]

SOLUTION I.

The equation of the parabola is

$$\frac{l}{\rho} = 1 - \cos\vartheta.$$

Let a, α be the co-ordinates of the centre of the circle; its equation will be

$$\rho = 2a\cos(\vartheta - \alpha).$$

For common points

$$\rho = \frac{2a}{\rho}\left[(\rho - l)\cos\alpha + \sin\alpha\sqrt{(2\rho l - l^2)}\right].$$

Clear of fractions, transpose, and square, and we have a quartic in ρ, of which the absolute term is $4a^2l^2$;

$$\therefore \rho_1\rho_2\rho_3\rho_4 = 4a^2l^2,$$

and is independent of α. [*Chas. Puryear.*]

[Prof. Graves directs attention to the fact that this property is given in Smith's *Conic Sections*. *Pereant qui nostra ante nos dixerunt!* By an analagous proof it is easy to show that the relation holds for all conics. Prof. Bohannan gives the same proof as above, and shows that the sum of the reciprocals of the focal radii is also constant.—*W. M. T.*]

SOLUTION II.

The equation of a parabola, referred to the axis and directrix, is

$$y^2 = 4p(x - p). \qquad (1)$$

The equation of a circle, radius r, through the focus $(2p, 0)$ is

$$(x-a)^2+(y-b)^2=(2p-a)^2+b^2=r^2. \tag{2}$$

From (1) and (2) we find

$$x^2+2(2p-a)x-4p(2p-a)=2by. \tag{3}$$

Squaring and eliminating y^2,

$$x^4+4(2p-a)x^3-4a(2p-a)x^2-16pr^2+16p^2r^2=0. \tag{4}$$

If x_1, x_2, x_3, x_4 are the roots of equation (4), the product of focal radii to the intersections of (1) and (2) is

$$x_1x_2x_3x_4=16p^2r^2, \tag{5}$$

which is constant, if p and r are constant.

COROLLARY. We also have

$$x_1x_2x_3+x_2x_3x_4+x_3x_4x_1+x_4x_1x_2=16pr^2. \tag{6}$$

Dividing by (5)

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=\frac{1}{p}. \tag{7}$$

Therefore, if circles be drawn through the focus of a fixed parabola cutting the curve in four points, the sum of the reciprocals of the focal radii to these points is constant. [*Wm. E. Heal.*]

76

FIND the condition that the cubic

$$6x^3-(2n+8)x^2+n(n+1)x+n(n+1)(2-n)=0$$

may have equal roots

SOLUTION I.

The discriminant of $a_0x^3+3a_1x^2+3a_2x+a_3=0$ is known to be*

$$\begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix};$$

$$\begin{vmatrix} 18 & -4(n+4) & n(n+1) & 0 \\ 0 & 18 & -4(n+4) & n(n+1) \\ -2(n+4) & 2n(n+1) & 3n(n+1)(2-n) & 0 \\ 0 & -2(n+4) & 2n(n+1) & 3n(n+1)(2-n) \end{vmatrix}=0$$

is the required condition.

*See Burnside and Panton's *Theory of Equations*, p. 305. Eliminant of $a_0x^2+2a_1x+a_2=0$ and $a_1x^2+2a_2x+a_3=0$ (Dialytic Method).

Drop successively the factors n, $(n+1)$, $(n-2)$, $(n-1)$ along with other reductions, as follows: —

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -(n+4) & n(n+1) & 3n(n+1)(2-n) & 0 \\ 0 & -(n+4) & 2n(n+1) & 3(2-n) \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -(n+4) & n(n+1) & 3n(n+1)(2-n) & 0 \\ -18 & 3(n+4) & 0 & 3(2-n) \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -n+2 & n^2-4 & 3n(n+1)(2-n) & n-2 \\ -6 & n+4 & 0 & 2-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -1 & n+2 & -3n(n+1) & 1 \\ -6 & n+4 & 0 & 2-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 0 & 7n+10 & -26n(n+1) & 9 \\ 0 & 9 & -4(n+4) & 1 \\ -1 & n+2 & -3n(n+1) & 1 \\ 0 & -5n-8 & 18n(n+1) & -4-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n+10 & -13n(n+1) & 9 \\ 9 & -2(n+4) & 1 \\ -5n-8 & 9n(n+1) & -4-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+5n+72 & 0 \\ 9 & -2(n+4) & 1 \\ 4n+28 & 7n^2-7n-32 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+5n+72 \\ 4n+28 & 7n^2-7n-32 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+12n+1 \\ 4n+28 & 7n^2-3n-4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n-1 \\ 4n+28 & 7n+4 \end{vmatrix} = 0,$$

or $$101n^2 - 101n - 256 = 0;$$
therefore the required condition is $n = -1$, or $= 0$, or $= 1$, or $= 2$; or $101n^2 - 101n - 256 = 0$.

The last condition, however, would give to the cubic irrational co-efficients.

[*R. H. Graves.*]

SOLUTION II.

The cubic $$Ax^3 + Bx^2 + Cx + D = 0$$
will have equal roots, if its discriminant

$$\begin{aligned} &27A^2D^2 + 4AC^3 + 4DB^3 - B^2C^2 - 18ABCD \\ &= D(27A^2D + 4B^3 - 18ABC) + 4AC^3 - B^2C^2 \end{aligned}$$

vanish. In the given equation $4AC^3 - B^2C^2$ is easily seen to be divisible by D, and therefore one condition of equal roots is

$$D = 0,$$

which is satisfied by $n = 0$, $n = -1$, $n = 2$.

Dividing by D and equating the remaining factor to zero, we have

$$101n^3 - 202n^2 - 155n + 256 = 0.$$

One root is evidently $n = 1$, and we easily find the others to be

$$n = \tfrac{1}{2} + \tfrac{15}{202}\sqrt{505}, \quad n = \tfrac{1}{2} - \tfrac{15}{202}\sqrt{505}.$$

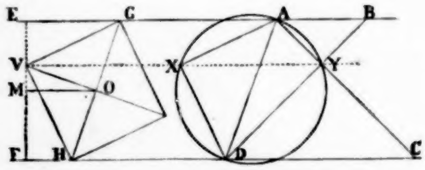
[*Wm. E. Heal.*]

77

Construct a square; given one vertex and two parallel lines on which the extremities of the opposite diagonal are located.

SOLUTION I.

Let V be the vertex, and AB, CD the parallel lines given. Let fall from V on AB, CD the perpendiculars VE, VF. On the same side of EF, when V is between the parallels, otherwise on opposite sides, on AB lay off $EG = VF$, and on CD lay off $FH = VE$. G and H will be the extremities of the opposite diagonal, whence the square may be easily constructed.



[*Ormond Stone.*]

[Solved in the same way substantially by R. D. Bohannan and Chas. Puryear.]

SOLUTION II.

Through the given point V draw a straight line VX, parallel to the given lines AB, CD. At any point Y, of VX, draw two lines YAC, YBD, each making an angle of $45°$ with VX. Through AYD pass a circle cutting VX in X. Then

$AX = DX$ and they are at right angles to each other. Lines drawn through V parallel to AX, DX will be sides of the required square.

[*R. D. Bohannan.*]

SOLUTION III.

Draw VM perpendicular to the parallel midway between the given parallels, and on the latter lay off MO equal to half the distance between the given parallels. GOH perpendicular to OV is the required diagonal.

[*W. M. Thornton.*]

78

GENERALIZATION.

A, B, C, etc. are the points of application of a system of parallel forces x, y, z, etc.; Q is their centre; ρ is their resultant; p, q, r, etc. are the radius vectors of A, B, C, etc., measured from an arbitrary origin M; $u = AB$, etc.

Show that if $MQ = h$

$$h^2 = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma u^2 xy}{\rho^2}.$$

SOLUTION.

Let $d_1 = QA$, $d_2 = QB$, etc.

It may be shown, as in articles 138 and 139 of Todhunter's *Analytical Statics*, that

$$\Sigma p^2 x = h^2 \rho + \Sigma d_1^2 x \quad \text{and} \quad \rho \Sigma d_1^2 x = \Sigma u^2 xy;$$

$$\therefore \; h^2 = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma d_1^2 x}{\rho} = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma u^2 xy}{\rho^2}.$$

[*R. H. Graves.*]

82

THE major axes of two similar and equal concentric ellipses intersect at right angles, and the area common to the two curves is half that of either ellipse. Find the eccentricity. [*Ormond Stone.*]

SOLUTION.

The conditions of the problem give the equation,

$$\tfrac{1}{2}a^2 b^2 \int_0^{\frac{1}{4}\pi} \frac{d\vartheta}{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta} = \tfrac{3}{16}\pi ab,$$

in which the usual notation is employed. Whence, by reduction,

$$b / a = \tan \tfrac{1}{8}\pi = \sqrt{2} - 1;$$

$$\therefore \; e = \sqrt{\left(\frac{a^2 - b^2}{a^2}\right)} = \sqrt{[2(\sqrt{2} - 1)]}.$$

[*R. H. Graves.*]

EXERCISES.

93

FIND the radius of the circle inscribed in the evolute of an ellipse and the ratio in which each point of contact divides the quadrant on which it lies.

[*R. H. Graves.*]

94

O is the centre of the circumscribed circle of ABC, and D, E, F the middle points of its sides. Show that

$$OD^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

95

IN exercise 65 let k, l be the lengths of the perpendiculars CD, CE drawn at right angles to CA, CB to meet the base in D, E. Show that

$$\frac{m}{c} = \frac{k}{a} \cdot \frac{l}{b}.$$

[*Ormond Stone.*]

96

GIVEN on the ground a circular curve of known radius intersecting a given straight line at a given point, and inclined to it at that point at a given angle; it is required to determine the radius of a second circular arc which shall be tangent both to the given curve and to the given line at another given point.

[*Calvin Whiteley.*]

97

IN the triangle ABC two lines drawn from C trisect the side AB. Given c, C, and the angle φ between the trisecants; to solve the triangle.

[*Marcus Baker.*]

98

THE eccentric anomalies of three points on an ellipse are p_1, p_2, p_3. Show that the area of their triangle is

$$\Delta = 2ab \sin\frac{p_2 - p_3}{2} \sin\frac{p_3 - p_1}{2} \sin\frac{p_1 - p_2}{2};$$

the centre of its circumscribing circle

$$x = +\frac{c^2}{a} \cos\frac{p_2 + p_3}{2} \cos\frac{p_3 + p_1}{2} \cos\frac{p_1 + p_2}{2},$$

$$y = -\frac{c^2}{b} \sin\frac{p_2 + p_3}{2} \sin\frac{p_3 + p_1}{2} \sin\frac{p_1 + p_2}{2};$$

and hence show that the centre of curvature of the ellipse at (x, y) is

$$X = +\frac{c^2x^3}{a^4}, \quad Y = -\frac{c^2y^3}{b^4}.$$

[*Wm. M. Thornton.*]

99

FROM a full cask of wine a quantity is taken out at random and the cask filled with water, and then a quantity of the mixture is taken out at random and the cask again filled with water. What is the probability that the cask now contains more wine than water? [*Artemas Martin.*]

SELECTED.

100

THE points O, O' defined by the equations in trilinears

$$a\alpha : b\beta : c\gamma = c^2a^2 : a^2b^2 : b^2c^2,$$
$$a\alpha : b\beta : c\gamma = a^2b^2 : b^2c^2 : c^2a^2,$$

are called the Brocard points. The angles OCA, OAB, OBC, $O'BA$, $O'CB$, $O'AC$ are called the Brocard angles.

1. Show that the Brocard angles are all equal each to

$$\cot^{-1}[\cot A + \cot B + \cot C].$$

2. Find the equation to the Brocard line OO'.

3. Find the equation to the Brocard circle through O, O' and the centre of the circumscribed circle.

4. Given the base BC and the Brocard angle of a triangle, find the locus of the vertex.

5. Show that the medians bisect the angles between the bisectrices of the angles of ABC and the "symmedian lines"

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c}.$$

6. Show that the Brocard circle contains the symmedian point.

101

EXPRESS in terms of $\sin^{-1}x$ and $\sin^{-1}y$

$$\tan^{-1}\frac{x+y}{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}.$$

102

FIND the relation connecting x, y, z when

$$\cot^{-1}(x+y+z-xyz) = \cot^{-1}x + \cot^{-1}y + \cot^{-1}z.$$

103

INTO a conical wine glass a spherical ball is dropped. Find the ratio of the concealed surfaces of the ball and the inside of the glass.

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ERRATA.

Vol. II. p. 36, lines 9, 10, and 11, read $d\varphi$ instead of $d\rho$.

Vol. II. p. 36, line 13, in second term of denominator in third term, read $\frac{e^2}{(d-r)^2}$ instead of $\frac{e}{(d-r)^2}$.